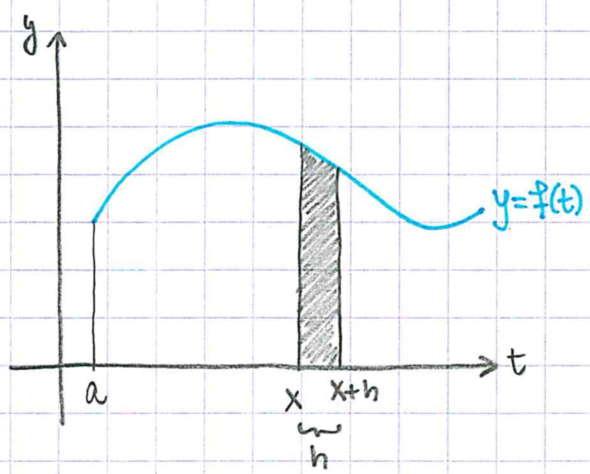
4.3. The Fundamental Theorem of Calculus



$$\boxed{g(x) = \int_a^x f(t)\,dt}$$

$$g(x+h) - g(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

$$= \underbrace{\int_x^{x+h} f(t)\,dt}_{\text{Area under graph of } f \text{ b/w } t=x \text{ and } t=x+h}$$

$\approx$ rectangle:

$$\int_x^{x+h} f(t)\,dt \approx h \cdot f(x)$$

$$\Rightarrow \frac{g(x+h)-g(x)}{h} \approx f(x) \quad \text{(when } h \text{ is very small)}$$

Intuition behind:

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f(x).$$

FTC $\boxed{\text{I}}$: If f is continuous on $[a,b]$, then the function g defined by:

$$g(x) := \int_a^x f(t)\,dt, \quad a \le x \le b$$

is continuous on $[a,b]$ and differentiable on (a,b), with $\boxed{g'(x) = f(x)}$

Example: If $g(x) = \int_1^x \frac{1}{1+t^5}\,dt$, what is $g'(x)$? $\boxed{g'(x) = \frac{1}{1+x^5}}$

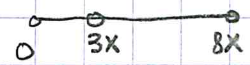
Example: $G(x) = \int_x^{20} \tan(4t^9)\,dt$; G' ? G'' ?

$$G(x) = -\int_{20}^x \tan(4t^9)\,dt \Rightarrow G'(x) = -\tan(4x^9)$$

$$G''(x) = ? \quad G''(x) = -\sec^2(4x^9)\cdot(4\cdot 9x^8) = -36x^8\sec^2(4x^9)$$

Example: $g(x) = \int_{3x}^{8x} \frac{3}{u^2+7}\,du$; $g'(x) = ?$

$$g(x) = \underbrace{\int_0^{8x} \frac{3}{u^2+7}\,du}_{} - \int_0^{3x} \frac{3}{u^2+7}\,du$$



Chain Rule!

$$\frac{d}{dx}\left(\int_0^{8x} \frac{3}{u^2+7}\,du\right) = \frac{3}{(8x)^2+7}\cdot 8$$

$$\frac{d}{dx}\left(\int_0^{3x} \frac{3}{u^2+7}\,du\right) = \frac{3}{(3x)^2+7}\cdot 3$$

$$\Rightarrow g'(x) = \frac{24}{(8x)^2+7} - \frac{9}{(3x)^2+7}$$

Example: $f(x) = \int_0^x \frac{t^2-9}{6+\cos^2(t)}\,dt$

ⓐ Critical numbers of f?

$f'(x) = \frac{x^2-9}{6+\cos^2(x)}$ $\Rightarrow$ c.no.'s: ± 3

ⓑ Sign of f'?

x		-3		3	
$f'(x)$	$+$	0	$-$	0	$+$
f	$\nearrow$	max	$\searrow$	min	$\nearrow$

FTC $\boxed{\text{II}}$ If f is continuous on $[a,b]$, then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

where F is <u>any</u> <u>antiderivative</u> of f.

Proof: $g(x) := \int_a^x f(t)\,dt \underset{\text{FTC 1}}{\Rightarrow} g'(x) = f(x)$

$\Rightarrow$ F and g differ by a constant:

$$\boxed{F(x) = g(x) + C} \rightarrow x=a: \; F(a) = g(a) + C = \underbrace{\int_a^a f(t)\,dt}_{0} + C$$

$$\Rightarrow \boxed{C = F(a)}$$

$$\Rightarrow \boxed{F(x) = g(x) + F(a)} \Rightarrow F(b) = g(b) + F(a)$$

$$\Rightarrow g(b) = \int_a^b f(t)\,dt = \underline{\underline{F(b) - F(a)}}.$$

Examples:

① $$\int_2^9 (5x^2 - 4x + 5)\,dx = \left(5\cdot\frac{x^3}{3} - 4\cdot\frac{x^2}{2} + 5x\right)\Big|_2^9$$
$$= \left(5\cdot\frac{9^3}{3} - 4\cdot\frac{9^2}{2} + 5\cdot 9\right) - \left(5\cdot\frac{2^3}{3} - 4\cdot\frac{2^2}{2} + 5\cdot 2\right)$$

② $$\int_1^{\sqrt{3}} \frac{14s^9 + 3\sqrt{s}}{s^9}\,ds = \int_1^{\sqrt{3}} \left(14 + 3s^{1/2-9}\right)ds = \int_1^{\sqrt{3}} \left(14 + 3s^{-17/2}\right)ds$$
$$= \left(14s + 3\cdot\frac{s^{-15/2}}{-15/2}\right)\Big|_1^{\sqrt{3}} = \left(14s - \frac{2}{5}s^{-15/2}\right)\Big|_1^{\sqrt{3}}$$
$$= \left(14\sqrt{3} - \frac{2}{5}(\sqrt{3})^{-15/2}\right) - \left(14 - \frac{2}{5}\right)$$

③ $$\int_{-5}^{-2} \left(\frac{6}{x^2} - 9\right)dx = \int_{-5}^{-2} (6x^{-2} - 9)\,dx = \left(6\,\frac{x^{-1}}{-1} - 9x\right)\Big|_{-5}^{-2}$$
$$= \left(\frac{-6}{x} - 9x\right)\Big|_{-5}^{-2} = \left(\frac{6}{2} + 18\right) - \left(\frac{6}{5} + 45\right)$$

④ $$\int_0^{\pi/4} 18\sec^2 x\,dx = 18\tan x\,\Big|_0^{\pi/4} = 18\left(\tan\pi/4 - \tan 0\right) = 18(1-0) = 18$$